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From Forsyth's Differential Equations pp. 192, 193, the values of y_1 and y_2 are merely his solutions I and IX, respectively, of the differential equation satisfied by $F(a, \beta, \gamma, x)$. But from p. 201, art. 128

$$y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} = Cx^{-\gamma} (1-x)^{\gamma-\alpha-\beta-1}.$$

Replacing x, a, β, γ , by their values in terms of z, m, n, p , and bringing down the factor z^{m-1} , we get, finally, omitting constant factors,

$$z^{n-mp-1} (z^m - 1)^{2p} = 0.$$

CAMBRIDGE, MASS., APRIL, 1899.

NOTE ON THE FUNCTION SATISFYING THE FUNCTIONAL RELATION $\phi(u) \cdot \phi(v) = \phi(u+v)$.*

BY E. B. WILSON.

The exponential function a^x ,

- 1) is defined for all values of the argument x ,
- 2) is single-valued,
- 3) satisfies the functional relation $\phi(u) \cdot \phi(v) = \phi(u+v)$.

Conversely any function which satisfies conditions 1, 2, 3 and is *continuous*, is the exponential function $y = [\phi(1)]^x$.†

The fundamental property on which the proof rests is that if x_0 be any value whatsoever and x any rational number, $\phi(x \cdot x_0) = [\phi(x_0)]^x$. This shows, to speak geometrically, that values of the variable commensurable with any particular value x_0 , give points lying on the curve $y = [\phi(x_0)]^x$. These points do not cover the curve completely, but they are everywhere dense. If the condition of continuity, which up to this point of the proof has not been used, be now added, the function is completely determined as $[\phi(1)]^x$.

Whether or not a *discontinuous* function may satisfy the relation $\phi(u) \cdot \phi(v) = \phi(u+v)$ has, so far as the author of this note is aware, never been decided. We shall however prove the

THEOREM: *If a function exists, which is defined for all values of the argument, is single-valued, satisfies the functional relation $\phi(u) \cdot \phi(v) = \phi(u+v)$,*

* This paper was presented to the American Mathematical Society at its meeting of April 29, 1899.

† For proof cf. Tannery: *Functions d'une Variable* §80, p. 120.

and has one discontinuity, then, in the neighborhood of any value x_0 of the argument, the function takes on values arbitrarily near to any preassigned positive value y_0 .

Proof: Form the auxiliary function

$$\psi(x) = \frac{\phi(x)^*}{[\phi(1)]^x}.$$

If x is rational $\phi(x) = \phi(x \cdot 1) = [\phi(1)]^x$ as above stated. Therefore $\psi(x) = 1$ when x is rational. Moreover, $\psi(u) \cdot \psi(v) = \psi(u+v)$. Hence if a and b be any two rational numbers $\psi(ax+b) = \psi(ax) \cdot \psi(b) = [\psi(x)]^a \cdot 1 = [\psi(x)]^a$.

Let x' be a value of x for which $\phi(x)$ is discontinuous. $\psi(x)$, which is the ratio of $\phi(x)$ to a continuous function $[\phi(1)]^x$ must also be discontinuous at x' . Hence in the neighborhood of x' there must be a value x'' of x such that $\psi(x'') \neq 1$. $x'' \neq 0$, for $\psi(0) = 1$.

Since $\psi(x'') \neq 1$,† obviously

1) a rational number a can be so chosen that

$$| [\psi(x'')]^a - c | < \gamma,$$

c, γ being any preassigned arbitrary positive numbers; and

2) a second rational number b can be so chosen that

$$| ax'' + b - x_0 | < \delta,$$

x_0 being any arbitrary preassigned number, and δ any arbitrary positive number. It has, however, been shown already that $[\psi(x'')]^a = \psi(ax'' + b)$. Hence $|\psi(ax'' + b) - c| < \gamma$, and $|ax'' + b - x_0| < \delta$.

To put these inequalities in words:—

In the neighborhood of any preassigned value x_0 of x , the function $\psi(x)$ will take on values arbitrarily near to any preassigned positive value. Or geometrically: Every two-dimensional region of the upper half plane contains points of the locus $y = \psi(x)$.

The equation $\phi(x) = \psi(x) \cdot [\phi(1)]^x$ connects $\psi(x)$ and $\phi(x)$. $[\phi(1)]^x$ is a continuous function which never takes on the value zero. Hence $\psi(x) \cdot [\phi(1)]^x$ will give, just as $\psi(x)$ did, points in every two-dimensional region of the upper half plane, i. e. $\psi(x) \cdot [\phi(1)]^x$ takes on values arbitrarily near to any preassigned positive value.

CAMBRIDGE, MASS., APRIL, 1899.

* $\phi(1) \neq 0$, as otherwise $\phi(1) \cdot \phi(-1) = \phi(0) = 0$; but $\phi(0) = 1$ cf. Tannery.

† $\psi(x'') > 0$, for $\phi(x) > 0$ cf. Tannery.